

# Quantum fluctuations of a “constant” gauge field

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It is argued here that the quantum computation of the vacuum pressure must take into account the contribution of zero-point oscillations of a rank-three gauge field. The field  $A_{\mu\nu\rho}$  possesses no radiative degrees of freedom, its sole function being that of polarizing the vacuum through the formation of *finite* domains characterized by a non-vanishing, constant, but otherwise arbitrary pressure. This extraordinary feature, rather unique among quantum fields, is exploited to associate the  $A_{\mu\nu\rho}$  field with the “bag constant” of the hadronic vacuum, or with the cosmological term in the cosmic case. We find that the quantum fluctuations of  $A_{\mu\nu\rho}$  are inversely proportional to the confinement volume and interpret the result as a Casimir effect for the hadronic vacuum. With these results in hands and by analogy with the electromagnetic and string case, we proceed to calculate the Wilson loop of the three-index potential coupled to a “test” relativistic bubble. From this calculation we extract the static potential between two opposite points on the surface of a spherical bag and find it to be proportional to the enclosed volume.

## I. INTRODUCTION

It is well known that the cosmological term introduced in General Relativity can be expressed as the vacuum expectation value of the energy-momentum tensor, as one might expect on the basis of relativistic covariance

$$\langle T^{\mu\nu} \rangle = \frac{\Lambda}{8\pi G} g^{\mu\nu} . \quad (1)$$

It is less well known that the same cosmological term can be formulated as the gauge theory of a rank-three antisymmetric tensor gauge potential  $A_{\mu\nu\rho}$  [1], [2], [3],[4] with an associated field strength

$$F_{\mu\nu\rho\sigma} = \nabla_{[\mu} A_{\nu\rho\sigma]} \quad (2)$$

invariant under the tensor gauge transformation

$$A_{\mu\nu\rho} \longrightarrow A_{\mu\nu\rho} + \nabla_{[\mu} \lambda_{\nu\rho]} . \quad (3)$$

Indeed, one readily verifies that the classical action

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{2 \times 4!} \int d^4x \sqrt{-g} F^{\lambda\mu\nu\rho} F_{\lambda\mu\nu\rho} \quad (4)$$

leads to the familiar Einstein equations in the presence of a cosmological term[4], [5]. Equation (1) suggests that the cosmological term is associated with the zero-point energy of the cosmic vacuum. Then, in view of the *equivalence* stated above, we are naturally led to question the calculability of the zero-point energy due to the quantum fluctuations of the  $A_{\mu\nu\rho}$ -field. At first sight, this may seem as an exercise in futility since a *constant* background field, represented by the field strength (2), cannot propagate any physical degree of freedom. However, we shall argue in the following sections that there are non-trivial volume effects due to the quantum fluctuations of the  $A$ -field.

Let us switch now from the cosmological case to the hadronic case and consider the implications of quantum vacuum energy in connection with the outstanding problem of color confinement in the theory of strong interactions. *Somewhat*

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surprisingly, perhaps, the formal connection between the two extreme cases, cosmological and hadronic, is provided by the same three-index potential  $A_{\mu\nu\rho}$  introduced earlier. Quantum Chromodynamics is universally accepted as the fundamental gauge theory of quarks and gluons. Equally accepted, however, is the view that  $QCD$  is still poorly understood in the non perturbative regime where the problem of color confinement sets in.

On the other hand, the phenomenon of quark confinement is accounted for, as an input, by the phenomenological “bag models,” with or without surface tension [6]. In some such models, it is assumed, for instance, that the normal vacuum is a color magnetic conductor characterized by an infinite value of the color magnetic permeability while the interior of the bag, even an empty one, is characterized by a finite color magnetic permeability. In the interior of the bag the vacuum energy density acts as a hadronic “cosmological constant” originating from zero-point energy due to quantum fluctuations inside the bag. This is a type of Casimir effect for the hadronic vacuum. To our knowledge, in spite of the fairly large amount of literature on the subject [7], this effect has never been discussed before in terms of the  $A_{\mu\nu\rho}$  field. Ultimately, the origin of this effect, and therefore of the cosmological bag constant should be traced back to the fundamental dynamics of the Yang-Mills field. Our suggestion, to be discussed in detail in a forthcoming publication, is that the link between the  $A_{\mu\nu\rho}$ -field and the fundamental variables of  $QCD$  is given by the “topological density”  $Tr \mathbf{F}^{\mu\nu} * \mathbf{F}_{\mu\nu}$  in  $QCD$  through the specific identification

$$A_{\mu\nu\rho} = \frac{1}{16\pi^2 \Lambda_{QCD}^2} Tr \left( \mathbf{A}_{[\mu} \partial_{\nu} \mathbf{A}_{\rho]} + \mathbf{A}_{[\mu} \mathbf{A}_{\nu} \mathbf{A}_{\rho]} \right) . \quad (5)$$

In support of this identification, notice that a Yang-Mills gauge transformation in Eq.(5) induces an *abelian* gauge transformation of the type (3)

$$\delta A_{\mu\nu\rho} = \frac{1}{g\Lambda_{QCD}} Tr \left[ \left( \mathbf{D}_{[\mu} \mathbf{A} \right) \mathbf{F}_{\nu\rho]} \right] \equiv \frac{1}{g} \partial_{[\mu} \lambda_{\nu\rho]} \quad (6)$$

where  $\Lambda_{QCD}$  is the energy scale at which  $QCD$  becomes intrinsically non-perturbative.

Against this background, this paper is the second in a series dealing with the hadronic and cosmological implications of the vacuum quantum energy associated with the three-index potential  $A_{\mu\nu\rho}$ . In view of the chain of arguments offered above, we shall refer to that field as the “cosmological field,” or “topological field” depending on the specific application under consideration. Some such applications in the cosmological case, in particular in connection with the problem of dark energy and dark matter in the universe have been discussed in our first paper of the series [5]. Rank three gauge potentials also appear in different sectors of high energy theoretical physics, e.g., supergravity [2], cosmology [8], both gauge theory of gravity [9] and of extended objects [10]. As argued above, a central role is played by this kind of gauge field in connection with the problem of confinement [11]. The present article focuses on the general properties of the topological field as an *abelian gauge field of higher rank* but with an eye on the future discussion of the problem of confinement in  $QCD$ . The main idea, here, is to lay the ground by preparing the tools, both conceptual and technical for that discussion. Ultimately, we wish to calculate the Wilson loop for the three-index potential associated with a bag with a boundary represented by the three-dimensional world history of a spherical bubble. To our knowledge, this calculation has never been done before and will pave the way to the future inclusion of fermions in the model. Our calculations are performed in the euclidean regime and represent a generalization of the more conventional calculations for the Wilson loop in the case of quantum chromodynamic strings leading to the so called “area law” that is taken as a signature of color confinement [12]. From the Wilson loop we extract the static potential between two antipodal points on the surface of the bag and find it to be proportional to the volume enclosed by the surface. This is consistent with the basic underlying idea of confinement that it would require an infinite amount of energy to separate the two points. This calculation is performed in Section.4. As a stepping stone toward that calculation, we investigate in Section.3 what amounts to the Casimir effect for the  $A_{\nu\rho\sigma}$  field. Section.2 discusses some of the unique properties of the  $A_{\mu\nu\rho}$  field that are manifest even at the classical level. Some concluding remarks are offered in Sect.5.

## II. CLASSICALLY “TRIVIAL” DYNAMICS

Rank-three potentials  $A_{\mu\nu\rho}(x)$  were introduced as a generalization of the electromagnetic potential and of the Kalb-Ramond potential in string theory [13], [14], [15]. In the free case, i.e., when there is no interaction with “matter”, the classical dynamics described by the lagrangian density

$$L_0 \equiv \frac{1}{2 \cdot 4!} \left( \partial_{[\mu} A_{\nu\rho\sigma]} \right)^2 , \quad (7)$$

is *exactly solvable*: the field strength  $F_{\mu\nu\rho\sigma} \equiv \partial_{[\mu} A_{\nu\rho\sigma]}$  that solves the generalized Maxwell equations

$$\partial_\mu F^{\mu\nu\rho\sigma} = 0 \quad (8)$$

describes a constant background field,  $F_{\mu\nu\rho\sigma} = f \epsilon_{\mu\nu\rho\sigma}$ , where  $f$  is an arbitrary integration constant. In the absence of gravity, such a *classical* constant background field has no observable effects and can be rescaled to zero. At the quantum level, we argue in the following, this last statement requires some qualifications. In any case, the physical meaning of this background field becomes transparent when the  $A_{\mu\nu\rho}(x)$  potential is coupled to a rank-three current density  $J^{\mu\nu\rho}(x)$  with support over the spacetime history of a relativistic membrane, or 2-brane, [8] ( for later convenience, in this paper we work with euclidean, or Wick rotated, quantities )

$$\begin{aligned} L &= \frac{1}{2 \cdot 4!} \left( \partial_{[\mu} A_{\nu\rho\sigma]} \right)^2 - \frac{\kappa}{3!} J^{\mu\nu\rho} A_{\mu\nu\rho} \\ &= -\frac{1}{2 \cdot 4!} F^{\lambda\mu\nu\rho} F_{\lambda\mu\nu\rho} + \frac{1}{4!} F^{\lambda\mu\nu\rho} \partial_{[\lambda} A_{\mu\nu\rho]} - \frac{\kappa}{3!} J^{\mu\nu\rho} A_{\mu\nu\rho} \end{aligned} \quad (9)$$

$$\begin{aligned} J^{\mu\nu\rho}(x; Y) &\equiv \int_H \delta[x - Y] dY^\mu \wedge dY^\nu \wedge dY^\rho \\ &= \int_\Sigma d^3\sigma \delta^4[x - Y] \epsilon^{mnr} \partial_m Y^\mu \partial_n Y^\nu \partial_r Y^\rho \end{aligned} \quad (10)$$

where  $H$  is the target spacetime image of the world-manifold  $\Sigma$  through the embedding  $Y : \Sigma \longrightarrow H$ . In the *first order* formulation,  $F_{\lambda\mu\nu\rho}$  and  $A_{\mu\nu\rho}$  are treated as independent variables [16]. However, the  $F$ -field equation is algebraic rather than differential, and this provides the link between first and second order formulation:

$$\frac{\delta L}{\delta F^{\lambda\mu\nu\rho}} = 0 \longrightarrow F_{\lambda\mu\nu\rho} = \partial_{[\lambda} A_{\mu\nu\rho]} \quad (11)$$

$$\frac{\delta L}{\delta A^{\mu\nu\rho}} = 0 \longrightarrow \partial_\lambda F^{\lambda\mu\nu\rho} = \kappa J^{\mu\nu\rho}(x) . \quad (12)$$

The model lagrangian, Eq.(9), is the basis for classical and quantum “membrane dynamics”, CMD and QMD respectively. Provided that the current is divergence free, the model is invariant under extended gauge transformations:

$$\delta A_{\mu\nu\rho} = \partial_{[\mu} \lambda_{\nu\rho]} \longleftrightarrow \partial_\mu J^{\mu\nu\rho}(x) = 0 . \quad (13)$$

The divergence free condition (13) is satisfied whenever the membrane history has no boundary, which means either: (i) spatially closed, real membranes, whose world-track is infinitely extended along the timelike direction, or (ii) spatially closed, virtual branes emerging from the vacuum and recollapsing into the vacuum after a finite interval of proper time [17].

To prove that this is the case, let us compute the divergence of the current:

$$\begin{aligned} \partial_\mu J^{\mu\nu\rho}(x) &= \int_\Sigma d^3\sigma \left( \frac{\partial}{\partial x^\mu} \delta^4[x - Y] \right) \epsilon^{mnr} \partial_m Y^\mu \partial_n Y^\nu \partial_r Y^\rho \\ &= \int_\Sigma d^3\sigma \left( \frac{\partial}{\partial Y^\mu} \delta^4[x - Y] \right) \epsilon^{mnr} \partial_m Y^\mu \partial_n Y^\nu \partial_r Y^\rho \\ &= \int_\Sigma d^3\sigma \left( \partial_m \delta^4[x - Y] \right) \epsilon^{mnr} \partial_n Y^\nu \partial_r Y^\rho \\ &= \int_\Sigma d^3\sigma \epsilon^{mnr} \partial_m \left( \delta^4[x - Y] \partial_n Y^\nu \partial_r Y^\rho \right) \\ &= \int_H d \left( \delta^4[x - Y] dY^\nu \wedge dY^\rho \right) \\ &= \int_{\partial H = \emptyset} \delta^4[x - y] dy^\nu \wedge dy^\rho = 0 \end{aligned} \quad (14)$$

Thus,  $\partial_\mu J^{\mu\nu\rho}(x) = 0 \longleftrightarrow \partial H = \emptyset$ .

If  $J$  is divergence free, it can be written as the divergence of a rank four antisymmetric *bag current*  $K$

$$J^{\mu\nu\rho}(x) \equiv \partial_\lambda K^{\lambda\mu\nu\rho} \quad (15)$$

where

$$K^{\lambda\mu\nu\rho}(x) \equiv \int_B \delta^4 [x - z] dz^\lambda \wedge dz^\mu \wedge dz^\nu \wedge dz^\rho \quad (16)$$

and  $H \equiv \partial B$ . On the other hand,

$$dz^\lambda \wedge dz^\mu \wedge dz^\nu \wedge dz^\rho = \epsilon^{\lambda\mu\nu\rho} d^4 z, \quad (17)$$

so that one can write  $K^{\lambda\mu\nu\rho}(x)$  as

$$K^{\lambda\mu\nu\rho}(x) = \epsilon^{\lambda\mu\nu\rho} \Theta_B(x) \quad (18)$$

where

$$\Theta_B(x) = \int_B d^4 z \delta^4 [x - z] \quad (19)$$

is the *characteristic function* of the  $B$  manifold, i.e., a generalized unit step-function:  $\Theta_B(P \in B) = 1$ ,  $\Theta_B(P \notin B) = 0$ .

One can also express the bulk-current  $K$  in terms of the boundary current  $J$  by inverting Eq.(15):

$$\partial_\lambda K^{\lambda\mu\nu\rho} = J^{\mu\nu\rho}(x) \longrightarrow K^{\lambda\mu\nu\rho} = \partial^{[\lambda} \frac{1}{\partial^2} J^{\mu\nu\rho]} . \quad (20)$$

Now, by solving the Maxwell field equation (12), one finds the following equivalent forms of the classical  $F$  field

$$\begin{aligned} F^{\lambda\mu\nu\rho} &= f \epsilon^{\lambda\mu\nu\rho} + \kappa \partial^{[\lambda} \frac{1}{\partial^2} J^{\mu\nu\rho]} \\ &= \epsilon^{\lambda\mu\nu\rho} (f + \kappa \Theta_B(x)) \end{aligned} \quad (21)$$

where  $f$  is, again, the constant solution of the homogeneous equation. The presence of the membrane separates spacetime into two regions characterized by a different value of the energy density and pressure on either side of the domain-wall [8]. Thus, the  $A$  field produces at most a (constant) pressure difference between the interior and exterior of a closed 2-brane. However, this special static effect makes the  $A$ -field a very suitable candidate for providing a gauge description of the cosmological constant both in classical and quantum gravity.

On the other hand, as we have argued in the Introduction, it is quite possible that the phenomenon of color confinement in Quantum Chromodynamics is due to the abelian part of the Yang-Mills field and that the long-distance behavior of  $QCD$  can be effectively described in terms of the rank-three gauge potential (5) associated with the Yang-Mills topological density [11],[18]. Be that as it may, a bag model type of confinement mechanism can be obtained by coupling  $A_{\mu\nu\rho}$  to a membrane current density of the type (10) with support on the hadronic bag boundary. That this is the case may be argued even at the classical level [1]. However, our immediate objective here is to link the quark bag model mechanism of confinement directly to the *quantum* properties of  $A_{\mu\nu\rho}$  in a finite (four) volume.

### III. VACUUM FLUCTUATIONS AND HADRONIC CASIMIR PRESSURE

In view of our future discussion here and in subsequent articles the message of this section needs to be as clear as possible, so we state it at the outset and reiterate it now and then throughout this section. Suppose there is no closed 2-brane coupling with  $A$ . Then, the free field describes a non-vanishing background energy associated with the constant field strength  $f$  defined in (21) with  $\kappa = 0$ . Perhaps, this is most simply understood, physically, in terms of the energy momentum tensor derived from Eq.(4) in the limit of flat spacetime.

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}|_{g=\delta} \longrightarrow \frac{1}{3!} F_{\mu\alpha\beta\gamma} F_{\nu}^{\alpha\beta\gamma} - \frac{1}{2 \cdot 4!} \delta_{\mu\nu} F^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} \quad (22)$$

From here it follows that

$$T_{\mu\nu} = \frac{f^2}{2} \delta_{\mu\nu} . \quad (23)$$

At first sight, quantizing  $A$  seems to be meaningless because there are no dynamical degrees of freedom carried by  $A$ . Against this common misconception we argue that the *quantum dynamics* of  $A$  is non-trivial even in the free case since a consistent quantization of a “constant field” introduces a sort of *volume dynamics*. This is best understood in the “sum over histories approach” where we have to sum over all possible (constant, in our case) configurations of the field, and weigh each of them with the usual factor, namely,  $\exp(-\text{euclidean action})$ . The euclidean action is the four volume integral of the lagrangian density evaluated on the given field configuration. In the case of the  $A$ -field, the lagrangian density turns out to be constant over all possible configurations, and the euclidean action is simply: *euclidean action* = (*four volume*) $\times$  *constant*. Then, in the limit  $V \rightarrow \infty$  all quantum fluctuations are frozen and the value  $f = 0$  is singled out, as one might reasonably expect in the classical limit. By reversing the argument, *at the quantum level the  $A$ -field can assume a non vanishing, constant field strength  $F$ , only inside a finite volume space(time) region*. Even if non-dynamical in the usual sense,  $A_{\mu\nu\rho}$  plays an active role anyway: rather than propagating energy waves, or physical quanta, through spacetime, it “*digs holes into the vacuum*”. This unique behavior makes  $A$  the most appropriate candidate for describing a “bubbling vacuum” in which domains with different vacuum pressure endlessly fluctuate in and out of existence. Mathematically, the above picture of fluctuating virtual bubbles can be substantiated in terms of the “*finite volume*” partition functional  $Z(V)$

$$Z(V) = \int [dF] [DA] \exp[-S_0(F, A)] \quad (24)$$

$$S_0(F, A) = \int_V d^4x \left[ \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 - \frac{1}{4!} F^{\lambda\mu\nu\rho} \partial_{[\lambda} A_{\mu\nu\rho]} \right] . \quad (25)$$

At this stage,  $V$  represents the characteristic volume of the homogeneous fluctuations of the  $A$ -field. Later we shall discuss the case in which the spacetime region where fluctuations take place is bounded by a closed membrane coupled to  $A$ . This whole approach is reminiscent of the Casimir effect for the hadronic vacuum, a case-study that has been already widely reported in the literature [7]. The novelty of our approach consists in the use of the three-index gauge potential, which, to our knowledge, has never been considered before in connection with the Casimir effect. The main difference lies in the fact that, since the  $F$ -field is constant within the region of confinement, it is insensitive to the shape of the boundary, so that the resulting Casimir energy density and pressure are also independent of the shape of the boundary and are affected only by the size of the volume enclosed. In order to substantiate this statement, let us now turn to the technical side of our computation.

Let us start the calculation of  $Z(V)$  from the  $A$ -integration. The  $A_{\mu\nu\rho}$  integration measure includes gauge fixing and Fadeev-Popov ghosts that we will discuss in a short while. Before addressing this problem, it is worth observing that the action  $S_0$  can also be written in the form

$$S_0(F, A) = \int_V d^4x \left[ \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 + \frac{1}{3!} A_{\mu\nu\rho} \partial_\lambda F^{\lambda\mu\nu\rho} + \frac{1}{3!} \partial_\lambda (A_{\mu\nu\rho} F^{\lambda\mu\nu\rho}) \right] . \quad (26)$$

In order to avoid surface terms coming from the total divergence in Eq.(26), we assume that the volume of quantization has no boundary, for instance it is a four sphere.

The action  $S_0$  is invariant under the gauge transformation

$$\delta_\lambda A_{\mu\nu\rho} = \partial_{[\mu} \lambda_{\nu\rho]} \quad (27)$$

$$\delta_\lambda F_{\mu\nu\rho\sigma} = 0 \quad (28)$$

and the integration measure over  $A$  has to be properly defined in order to avoid over counting of physically equivalent field configurations. In the second order formulation, gauge invariance prevents one from inverting the kinetic operator and from computing the  $A$ -path integral ( in spite of its gaussian looking form ). The usual procedure is to break gauge invariance “by hand” and compensate the unphysical degrees of freedom produced by gauge fixing by means of an appropriate set of ghost fields. In the Lorentz gauge one finds

$$[DA] = [dA] \delta [\partial_\mu A^{\mu\nu\rho}] \Delta_{FP} \quad (29)$$

where the Fadeev-Popov determinant is defined through the gauge variation of the gauge fixing function

$$\begin{aligned} \Delta_{FP} &\equiv \det \left[ \frac{\delta}{\delta \lambda_{\mu\nu}} \partial^\rho \partial_{[\rho} \lambda_{\sigma\tau]} \right] \\ &= \det \left[ \partial^\rho \partial_{[\rho} \delta_\sigma^\mu \delta_\tau^\nu] \right] . \end{aligned} \quad (30)$$

The Fadeev-Popov procedure introduces a new gauge invariance which must in turn be broken and compensated until all the unphysical degrees of freedom are removed [19]. This lengthy procedure is necessary in order to perform perturbative calculations and compute Feynman graphs. However, we are interested in a non-perturbative evaluation of the path integral. With this goal in mind, let us remark that in the first order formulation  $A_{\mu\nu\rho}$  enters linearly into the action rather than quadratically. In other words, the non dynamical nature of  $A_{\mu\nu\rho}$  is made manifest in the first order formulation, where  $A_{\mu\nu\rho}$  plays the role of a Lagrange multiplier enforcing the classical field equation for  $F_{\lambda\mu\nu\rho}$ . Thus, instead of going through all the steps of the Fadeev-Popov procedure, we split  $A_{\mu\nu\rho}$  into the sum of a Goldstone term  $\theta_{\nu\rho}$  and a gauge inert part ( modulo a shift by a constant )  $\epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi$  [21]

$$A_{\mu\nu\rho} \equiv \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi + \partial_{[\mu} \theta_{\nu\rho]} \quad (31)$$

$$\delta_\lambda \phi = 0 , \quad \delta_\lambda \theta_{\nu\rho} = \lambda_{\nu\rho} . \quad (32)$$

Accordingly, the functional integration measure becomes

$$[dA] = J [d\phi] [d\theta] \quad (33)$$

where  $J$  is the functional Jacobi determinant induced by the change of integration variables (31) not to be confused with the Fadeev-Popov determinant.  $J$  reads

$$\begin{aligned} J &= [\text{Det} (-\partial^2)]^{1/2} \times \left[ \text{Det} \left( -\frac{1}{3!} \epsilon_{\sigma\alpha\beta\gamma} \partial^\gamma \epsilon^{\sigma\alpha\beta\rho} \partial_\rho \right) \right]^{1/2} \\ &= [\text{Det} (-\partial^2)] \end{aligned} \quad (34)$$

and provides the correct counting of the physical degrees of freedom. Apparently we introduced two new degrees of freedom:  $\theta$  and  $\phi$  while from the classical analysis we expect  $A$  to describe a constant background. Let us show first as  $\theta$  drops out form the path integral.

The classical action is  $\theta$  independent because of gauge invariance

$$S_0 (F, A) \equiv S_0 (F, \phi) \quad (35)$$

and does not provide the necessary damping of gauge equivalent paths. However, the gauge fixed-compensated integration measure reads

$$[DA] \equiv [d\phi] [d\theta] J \delta [\partial_\mu \partial^{[\mu} \theta^{\nu\rho]}] \Delta_{FP} \quad (36)$$

and we can get rid of the gauge orbit volume. Since

$$\int [d\theta] \delta [\partial_\mu \partial^{[\mu} \theta^{\nu\rho]}] \Delta_{FP} = 1 \quad (37)$$

we obtain a path integral over gauge invariant degrees of freedom only:

$$Z(V) = \int [dF] [d\phi] J \exp[-S_0(F, \phi)]$$

$$S_0(F, \phi) \equiv \int_V d^4x \left[ \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 - \frac{1}{3!} \partial_\lambda F^{\lambda\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi \right]. \quad (38)$$

Suppose we first integrate over  $F$ . This is a gaussian integration and we get

$$Z = \int [d\phi] J \exp \left[ - \int d^4x \frac{1}{2} \phi (-\partial^2) (-\partial^2) \phi \right] \quad (39)$$

Now, if we integrate over the scalar field  $\phi$  we see that the contribution of the  $\phi$  field fluctuations exactly cancel the Jacobian because of the “box squared” kinetic term:

$$Z = [\text{Det}(-\partial^2)] \times [\text{Det}(-\partial^2)^2]^{-1/2} = "1" \quad (40)$$

where, the quotation marks is a reminder to the presence of an everywhere understood global normalization constant. Thus, no spurious degrees of freedom have been introduced through (31). On the other hand, it is interesting to reverse the order of integration and start with  $\phi$  instead of  $F$ . In this case it is more convenient to introduce the new integration variable

$$U_{\mu\nu\rho} \equiv \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi \quad (41)$$

and write the integration measure as

$$[d\phi] = [dU] [\text{Det}(-\partial^2)]^{-1/2} = J^{-1/2} [dU] \quad (42)$$

Hence, we obtain

$$Z = \int [dU] [dF] J^{1/2} \exp \left[ - \int_V d^4x \left( -\frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 - \frac{1}{3!} \partial_\lambda F^{\lambda\mu\nu\rho} U_{\mu\nu\rho} \right) \right] \quad (43)$$

We notice that the path-integral is linear in the  $U$  variable. To integrate over this variable it is convenient to rotate, *momentarily*, from euclidean to minkowskian signature in such a way to reproduce a path-integral form of the Dirac delta-function

$$\int [dU] \exp \left( -\frac{i}{3!} \int_V d^4x U_{\mu\nu\rho} \partial_\lambda F^{\lambda\mu\nu\rho} \right) = \delta [\partial_\lambda F^{\lambda\mu\nu\rho}] \quad (44)$$

and then we rotate back to euclidean section. In such a way the calculation of  $Z(V)$  boils down to computing the path integral over the field strength configurations that satisfy the “constraint”  $\partial_\lambda F^{\lambda\mu\nu\rho} = 0$ :

$$Z(V) = \int [dF] J^{1/2} \delta [\partial_\lambda F^{\lambda\mu\nu\rho}] \exp \left[ - \int_V d^4x \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 \right]. \quad (45)$$

Notice that we are back to the euclidean signature. Since the constraint is nothing but the classical field equation satisfied by  $F$ , it is easy to implement it since in four dimensions the tensorial structure requires that  $F_{\lambda\mu\nu\rho} = F(x) \epsilon_{\lambda\mu\nu\rho}$ . Accordingly, all possible classical solutions are of the form  $F(x) = \text{const.} \equiv f$  where  $f$  is an arbitrary parameter. The path integral is then evaluated by replacing  $F$  with its constant value in the integrand ( and absorbing any field independent quantity in the global normalization constant ):

$$Z(V; f) = \exp \left[ -\frac{1}{2} f^2 V \right] \quad (46)$$

which is the standard result available in the literature [3]. Thus, the resulting partition function is vanishing in the limit  $V \rightarrow \infty$  for any value  $f \neq 0$ . In other words, the only allowed value is  $f = 0$  giving  $Z(V \rightarrow \infty) = "1"$ . This is the “trivial vacuum” corresponding to a vanishing energy density/pressure. However, when the volume is finite, one must take into account contributions from the *quantum vacuum fluctuations* of the  $F$ -field coming from all possible, constant, values of  $f$ . Here is where we depart from the conventional formulation of the sum over histories approach. Since  $f$  is constant but arbitrary, *the sum over histories amounts to integrating over all possible values of  $f$*

$$\begin{aligned} Z(V) &= \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \int [dF] J^{1/2} [\text{Det}(-\partial^2)]^{-1/2} \delta[F^{\lambda\mu\nu\rho} - f\epsilon^{\lambda\mu\nu\rho}] \exp\left[-\int_V d^4x \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2\right] \\ &= \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left[-\frac{1}{2} f^2 V\right] = \sqrt{\frac{2\pi}{V\mu_0^4}} \end{aligned} \quad (47)$$

where  $\mu_0$  is a fixed mass scale that is required in order to keep the integration measure dimensionless and all the Jacobian factors cancel. The final result is a “field independent constant” which is missing in the standard formulation. However, this “constant” keeps the memory of  $V$  which, in our case, represent the volume where the field fluctuations takes a non-vanishing value. Incidentally, this is the same technique that leads to the correct expression for the particle propagator in ordinary quantum mechanics [20].

From here we can proceed in two directions. First, we can calculate the size of the quantum fluctuations of the  $f$ -field; second, we can derive an expression for the vacuum energy density/pressure in the finite volume in which the quantum fluctuations of the  $f$ -field are confined.

With reference to the first point, since  $\Delta f$  is defined as

$$\Delta f \equiv \sqrt{\langle f^2 \rangle - \langle f \rangle^2} \quad (48)$$

we need to introduce an external source  $j$  in order to calculate the average values in Eq.(48). By definition

$$\langle f \rangle = - \left( \frac{1}{Z(f, j)} \frac{\partial Z(f, j)}{\partial j} \right)_{j=0} \quad (49)$$

$$\langle f^2 \rangle = \left( \frac{1}{Z(f, j)} \frac{\partial^2 Z(f, j)}{\partial j^2} \right)_{j=0} \quad (50)$$

where we use the expression (47) in the presence of an external source

$$Z(V) \longrightarrow Z(V; j) = \int_{-\infty}^{\infty} \frac{df}{\mu_0^2} \exp\left[-\frac{1}{2} f^2 V - j f\right]. \quad (51)$$

Equations (49) and (50) lead to the following results

$$\langle f \rangle = 0 \quad (52)$$

$$\langle f^2 \rangle = \frac{1}{V} \quad (53)$$

so that the variance of  $f$ , Eq.(48), is given by

$$(\Delta f)^2 = \langle f^2 \rangle = \frac{1}{V}. \quad (54)$$

The average of the  $F$ -field turns out to be zero since opposite values of  $f$  are weighed equally in the partition function (47). However, the final result (54) confirms that the quantum fluctuations of the  $F$ -field are confined in a finite volume, with larger volumes being associated with smaller and smaller fluctuations.

Let us now turn back to the promised expression for the vacuum energy density/pressure. This follows from the usual definition



$$p \equiv -\frac{\partial}{\partial V} \ln Z(V) . \quad (55)$$

Once we compare it with the explicit expression (47), we find

$$p = \frac{1}{2V} = \frac{1}{2} \langle f^2 \rangle \quad (56)$$

which tells us that the Casimir pressure is generated solely by the quantum fluctuations of the  $F$ -field and is inversely proportional to the quantization volume  $V$ . Up until now the volume of confinement has been kept fixed and we have calculated the average values of the field  $F$  and pressure  $p$  inside  $V$ . At this point we would like to turn this procedure around and calculate the *average volume* corresponding to fluctuations with a preassigned vacuum pressure. Here we face a technical difficulty since  $Z(V)$  behaves as  $1/\sqrt{V}$  and is therefore non integrable for large values of the argument. In order to get around this difficulty we need to integrate  $Z(V)$  over all possible volumes with an appropriate weight factor that plays the role of an infrared cut-off

$$Z(\rho_0) \equiv \int_0^\infty dV e^{-\rho_0 V} Z(V) = \frac{\pi}{\mu_0^2} \sqrt{\frac{2}{\rho_0}} . \quad (57)$$

Using this result we calculate

$$\langle V \rangle \equiv -\frac{1}{Z(\rho_0)} \frac{\partial Z(\rho_0)}{\partial \rho_0} = \frac{1}{2\rho_0} \quad (58)$$

which illustrates the role of the infrared cutoff  $\rho_0$  and its physical interpretation as the pressure due to the phenomenological bag constant.

### A. Generating Functional

This subsection has a double purpose: the first is to study the vacuum expectation value of the energy-momentum tensor as a check on the calculation discussed above; our second purpose is to compare the quantum computation of  $\langle T_{\mu\nu} \rangle$  with its classical counterpart already discussed at the beginning of Sect.(3). In order to study vacuum expectation values we need to introduce an appropriate *external source* coupled to the selected operator and then compute the corresponding generating functional. In our problem the hadronic vacuum pressure and energy density can be extracted from the expectation value of the energy momentum tensor operator

$$\langle T_{\mu\nu} \rangle = \left\langle \frac{1}{3!} F_{\mu\alpha\beta\gamma} F_{\nu}^{\alpha\beta\gamma} - \frac{1}{2 \cdot 4!} \delta_{\mu\nu} F^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} \right\rangle \quad (59)$$

$T_{\mu\nu}$  being the “current” canonically conjugated to the metric tensor. Thus, we switch-on a non-trivial background metric  $g_{\mu\nu}(x)$

$$S_0 \longrightarrow \frac{1}{2 \cdot 4!} \int_V d^4x \sqrt{g} g^{\alpha\beta} g^{\mu\gamma} g^{\nu\sigma} g^{\rho\tau} F_{\alpha\mu\nu\rho} F_{\beta\gamma\sigma\tau} \quad (60)$$

where  $g \equiv \det g_{\mu\nu}(x)$ . The metric  $g_{\mu\nu}(x)$  plays the role of external source for  $T_{\mu\nu}$ , which means

$$\langle T_{\mu\nu} \rangle \equiv \left( \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}(x)} \ln Z[g; V] \right)_{g=\delta} . \quad (61)$$

Thus, the result is formally the same as in Eq.(47) except for the presence of  $\sqrt{g}$  in the expression of the volume:

$$Z[g; V] = \left( \frac{2\pi}{\mu_0^4 V[g]} \right)^{1/2} \quad (62)$$

$$V[g] = \int_V d^4x \sqrt{g}, \quad V[g = \delta] = V. \quad (63)$$

The same difficulty noted before, namely, the non-integrability of  $Z[g; V]$  for large  $V[g]$  requires that, in order to generate a classical background pressure in  $\langle T_{\mu\nu} \rangle$  we consider the regularized partition function

$$Z[g; V] \longrightarrow Z_{reg}[g; V] \equiv Z[g; V] \exp(-\rho_0 V[g]). \quad (64)$$

Our objective now is to calculate

$$\langle T_{\mu\nu} \rangle \equiv \left( \frac{2}{\sqrt{g}} \frac{1}{Z_{reg}[g; V]} \frac{\delta Z_{reg}[g; V]}{\delta g^{\mu\nu}(x)} \right)_{g=\delta}. \quad (65)$$

Since we have

$$\frac{\delta Z_{reg}[g; V]}{\delta g^{\mu\nu}(x)} = -\frac{1}{2\mu_0^2} \sqrt{\frac{2\pi}{V[g]}} \sqrt{g} g_{\mu\nu} \left( \rho_0 + \frac{1}{2V[g]} \right) \exp(-\rho_0 V) \quad (66)$$

combining equations (66), (64) with the definition (65) we finally obtain

$$\langle T_{\mu\nu} \rangle|_{g=\delta} = \left( \rho_0 + \frac{1}{2V} \right) \delta_{\mu\nu}. \quad (67)$$

This final expression of  $T_{\mu\nu}$  confirms the previous calculation of the vacuum pressure as consisting of the quantum Casimir pressure superimposed to the phenomenological background pressure represented by  $\rho_0$ . This concludes our discussion of the classical and quantum effects due to the three-index potential  $A_{\mu\nu\rho}$  in the absence of interactions. The coupling to a relativistic test bubble will be the subject of next Section.

#### IV. HADRONIC BAGS

In the previous sections we computed the partition function for the hadronic vacuum by summing over constant configurations of the  $F$ -field inside finite volume vacuum domains. The resulting picture is one of a “bubbling” ground state in which *virtual* bags quantum mechanically fluctuate.

In this section we wish to study the behaviour of a *real* test bubble immersed in the quantum vacuum characterized by the Casimir energy of the  $A_{\mu\nu\rho}$  field. To begin with, within the test bubble the  $F$ -field may attain any value as opposed to the exterior (infinite) region where its value is zero.

Mathematically, this new situation corresponds to taking as a new action

$$S_0 \longrightarrow S_0 + \frac{\kappa}{3!} \int d^4x A_{\mu\nu\rho} J^{\mu\nu\rho} \quad (68)$$

where  $J^{\mu\nu\rho}$  is given in Eq.(10).

The finite volume partition function now reads

$$Z(V; J) = \int [dF] [DA] \exp[-S(F, A)] \quad (69)$$

$$S(F, A) = \int_B d^4x \left[ \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2 - \frac{1}{4!} F^{\lambda\mu\nu\rho} \partial_{[\lambda} A_{\mu\nu\rho]} - \frac{\kappa}{3!} J^{\mu\nu\rho} A_{\mu\nu\rho} \right] \quad (70)$$

$$V = \int_B d^4x. \quad (71)$$

Once again, let us start the calculation of  $Z(V; J)$  from the  $A$ -integration. The only difference with respect to the previous case is that a bag is endowed with a non-vanishing boundary. In this case, the total divergence in Eq.(26) may include a surface term defined over  $\partial B$ . The most convenient boundary condition is to assume  $A$  to be a *pure gauge* on  $\partial B$

$$\begin{aligned} \frac{1}{3!} \int_B d^4x \partial_\lambda (A_{\mu\nu\rho} F^{\lambda\mu\nu\rho}) &= \frac{1}{4!} \int_{\partial B} d^3\sigma [\lambda \partial_\mu \lambda_{\nu\rho}] \hat{F}^{\lambda\mu\nu\rho} \\ &\equiv \omega(F, \partial B) \end{aligned} \quad (72)$$

where  $\hat{F}$  is the field induced on the boundary by  $F$ . Proceeding in the manner discussed in the previous subsection, we find

$$\begin{aligned} Z(V; J) &= \int [dF] \delta[\partial_\lambda F^{\lambda\mu\nu\rho} - \kappa J^{\mu\nu\rho}] \times \\ &\quad \exp\left[-\int_B d^4x \frac{1}{2 \cdot 4!} F_{\lambda\mu\nu\rho}^2\right] \exp[-\omega(F, \partial B)] . \end{aligned} \quad (73)$$

The surface term does not contribute to the calculation of  $Z(V; J)$ , after integration over  $F$ , because of Stoke's theorem, while the effect of the current is to shift the constant background value  $f$  to  $f + \kappa$  within the membrane. Thus,

$$\begin{aligned} \frac{1}{2 \times 4!} \int_B d^4x F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} &= \frac{1}{2 \times 4!} \int_B d^4x \left( \epsilon^{\mu\nu\rho\sigma} f - \kappa \partial^{[\lambda} \frac{1}{-\partial^2} J^{\mu\nu\rho]} \right)^2 \\ &= \frac{1}{2} \int_B d^4x f_{in}^2 - \frac{f\kappa}{4!} \epsilon_{\mu\nu\rho\sigma} \int d^4x \partial^{[\lambda} \frac{1}{-\partial^2} J^{\mu\nu\rho]} + \frac{\kappa^2}{2 \times 3!} \int_B d^4x \partial^{[\lambda} \frac{1}{-\partial^2} J^{\mu\nu\rho]} \partial_{[\lambda} \frac{1}{-\partial^2} J_{\mu\nu\rho]} \\ &= \frac{1}{2} V f_{in} [f_{in} - 2\kappa \Theta_B(x)] + \frac{\kappa^2}{2 \times 3!} \int d^4x J^{\mu\nu\rho} \frac{1}{-\partial^2} J_{\mu\nu\rho} . \end{aligned} \quad (74)$$

The final result is obtained after integrating out  $f_{in}$ :

$$Z(V; J) = \sqrt{\frac{2\pi}{\mu_0^4 V}} \exp\left\{\frac{\kappa^2}{2} V\right\} \exp\left(-\frac{\kappa^2}{2 \times 3!} \int_B d^4x J^{\mu\nu\rho} \frac{1}{-\partial^2} J_{\mu\nu\rho}\right) . \quad (75)$$

The above expression represents the basic generating functional in the interacting case. It will be used in the next subsection for the purpose of computing the Wilson loop of the  $A$ -field. Note finally, that for  $\kappa = 0$  the expression (75) reduces to the free case discussed previously, as it should be.

### A. Wilson factor and the static potential

In this section we assume that the hadronic manifold  $B$  extends indefinitely along the euclidean time direction and keep the coupling term between  $A_{\mu\nu\rho}$  and the boundary. Our objective is to determine the static potential between pairs of points situated on the boundary of the test bubble that we take to be a spherical two-surface of radius  $R$ . The evolution of the two-sphere in euclidean time is represented by an hyper-cylinder  $I \times S^{(2)}$ , where  $I$  is the interval  $0 \leq t^E \leq T$  of euclidean time  $t^E$ . On the two-surface  $S^{(2)}$  let us “mark” a pair of antipodal points and follow their (euclidean) time evolution. The two points move along parallel segments of total length  $T$ . The standard calculation of the static potential between charges moving along an elongated rectangular loop, turns, in the case under study, into the calculation of the Wilson “loop” along the hyper-cylinder  $I \times S^{(2)}$ . The rectangular path is now given by the two segments of length  $T$  and diameter  $2R$  of the sphere at  $t^E = 0$  and  $t^E = T$ . The corresponding static potential is given by the following generalized Wilson integral

$$V(R) \equiv - \lim_{T \rightarrow \infty} \frac{1}{T} \ln W[\partial B] . \quad (76)$$

The path integral calculation of  $W[\partial B]$  starts from the finite volume boundary functional, Eq.(75). The Wilson factor is defined as follows

$$W[\partial B] \equiv \langle \exp \left[ -\frac{\kappa}{3!} \int d^4x A_{\lambda\mu\nu} J^{\lambda\mu\nu} \right] \rangle = \frac{Z[V; J]}{Z[V; J=0]} \quad (77)$$

where  $V < \infty$  is understood and the limit  $V \rightarrow \infty$  (along the euclidean time direction) is performed at the end of the calculations.

In order to extract the static potential  $V(R)$ , we compute the double integral in (75) for the currents associated to a pair of antipodal points  $P$  and  $\bar{P}$

$$\begin{aligned} \int_B d^4x J^{\mu\nu\rho} \frac{1}{\partial^2} J_{\mu\nu\rho} &= \int_{\partial B} \int_{\partial B} dy^\mu \wedge dy^\nu \wedge dy^\rho \frac{1}{\partial^2} dy'_\mu \wedge dy'_\nu \wedge dy'_\rho = \\ &= \frac{1}{4\pi^2} \int_0^T d\tau \int_T^0 d\tau' \int_{S^{(2)}} d^2\sigma \int_{S^{(2)}} d^2\xi \times \\ &\quad y^{\mu\nu\rho}(\tau, \sigma) \frac{1}{[y(\tau, \sigma) - y(\tau', \xi)]^2} y_{\mu\nu\rho}(\tau', \xi) \delta^2[\xi - \sigma] \end{aligned}$$

where  $(\sigma^1, \sigma^2)$  and  $(\xi^1, \xi^2)$  are two independent sets of world coordinates on the  $S^{(2)}$  manifold. Furthermore, we have inserted the explicit form of the scalar Green function and have indicated by

$$y^{\mu\nu\rho} = \epsilon^{abc} \partial_a y^\mu \partial_b y^\nu \partial_c y^\rho \quad (78)$$

the ‘‘tangent elements’’ to the world history of the test bubble. The membrane world manifold is an hypercylinder with euclidean metric given, in polar coordinates, by

$$ds^2 = \gamma_{ab}(\sigma) d\sigma^a d\sigma^b = d\tau^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (79)$$

where  $0 \leq \phi \leq 2\pi$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq \tau \leq T$ . The embedding in target spacetime is obtained through the equations

$$y^1 = R \sin \theta \sin \phi \quad (80)$$

$$y^2 = R \sin \theta \cos \phi \quad (81)$$

$$y^3 = R \cos \theta \quad (82)$$

$$y^4 = \tau. \quad (83)$$

Then, with the above choice of coordinates, we find

$$y^{\mu\nu\rho} = \partial_{[\tau} y^\mu \partial_\theta y^\nu \partial_\phi] y^\rho \quad (84)$$

$$y^{ijk} \equiv 0$$

$$y^{4ij} = \partial_{[\theta} y^i \partial_\phi] y^j.$$

The explicit expression of the tangent elements  $y^{ijk}$  evaluated at the point  $P$  can be written as follows

$$\begin{aligned} y^{12}(\theta, \phi) &\equiv \partial_{[\theta} y^1 \partial_\phi] y^2 = -R^2 \cos \theta \sin \theta \\ y^{13}(\theta, \phi) &\equiv \partial_{[\theta} y^1 \partial_\phi] y^3 = R^2 \sin^2 \theta \cos \phi \\ y^{23}(\theta, \phi) &\equiv \partial_{[\theta} y^2 \partial_\phi] y^3 = -R^2 \sin^2 \theta \sin \phi. \end{aligned}$$

Then, for the antipodal point  $\bar{P}$  the same expressions become

$$\begin{aligned} y^{12}(\pi - \theta, \phi + \pi) &= R^2 \cos \theta \sin \theta \\ y^{13}(\pi - \theta, \phi + \pi) &= -R^2 \sin^2 \theta \cos \phi \\ y^{23}(\pi - \theta, \phi + \pi) &= R^2 \sin^2 \theta \sin \phi. \end{aligned}$$

From the above expressions, we explicitly calculate

$$\ln W[\partial B] = \frac{\kappa^2}{48\pi^2} \int_0^T d\tau \int_T^0 d\tau' \int_0^\pi d\theta \int_0^{2\pi} d\phi \times y^{ij}(\theta, \phi) \frac{1}{[y(\theta, \phi) - y(\pi - \theta, \phi + \pi)]^2} y_{ij}(\pi - \theta, \phi + \pi) \quad (85)$$

so that

$$\frac{1}{2} y^{ij}(\theta, \phi) y_{ij}(\pi - \theta, \phi + \pi) = -R^4 \sin^2 \theta \quad (86)$$

$$[y(\theta, \phi) - y(\pi - \theta, \phi + \pi)]^2 = (\tau - \tau')^2 + 4R^2. \quad (87)$$

Therefore the logarithm of  $W[\partial B]$  is

$$\ln W[\partial B] = -\frac{\kappa^2 R^4}{48} \int_0^T d\tau \int_0^T d\tau' \frac{1}{(\tau - \tau')^2 + 4R^2}. \quad (88)$$

We now proceed to calculate the double integral in Eq.(88):

$$\begin{aligned} \int_0^T d\tau \int_0^T d\tau' \frac{1}{(\tau - \tau')^2 + 4R^2} &= -\int_0^T d\tau \int_\tau^{\tau-T} du \frac{1}{u^2 + 4R^2}, \quad u \equiv \tau - \tau' \\ &= -\frac{1}{2R} \int_0^T d\tau \int_{\tau/2R}^{(\tau-T)/2R} dy \frac{1}{1 + y^2} \\ &= -\frac{1}{2R} \int_0^T d\tau \left[ \arctan\left(\frac{\tau-T}{2R}\right) - \arctan\left(\frac{\tau}{2R}\right) \right] \end{aligned} \quad (89)$$

$$\begin{aligned} \int_0^T d\tau \arctan\left(\frac{\tau-T}{2R}\right) &= \int_{-T}^0 ds \arctan\left(\frac{s}{2R}\right), \quad \tau - T \equiv s \\ &= -\int_0^T ds \arctan\left(\frac{s}{2R}\right), \quad s \rightarrow -s. \end{aligned} \quad (90)$$

Putting together equations (89) and (90) we obtain

$$\int_0^T d\tau \int_0^T d\tau' \frac{1}{(\tau - \tau')^2 + 4R^2} = \frac{1}{R} \int_0^T d\tau \arctan\left(\frac{\tau}{2R}\right) = \frac{T}{R} \arctan\left(\frac{T}{2R}\right) + 2R \ln\left(1 + \frac{T^2}{4R^2}\right) \quad (91)$$

which, on account of the definition (76), leads to the final result

$$V(R) \equiv -\lim_{T \rightarrow \infty} \frac{1}{T} \ln W[\partial B] = \frac{\pi \kappa^2}{96} R^3. \quad (92)$$

According to Eq.(92) the antipodal points on the spherical membrane of radius  $R$  are subject to an attractive potential varying with the volume enclosed by the membrane.

## V. CONCLUSIONS

In this paper we have tried to make a case that the hadronic vacuum represents an ideal laboratory to test a new approach to the quantum computation of the vacuum pressure in terms of an antisymmetric, rank-three, tensor gauge field  $A_{\mu\nu\rho}$  possibly realized in  $QCD$  by the collective excitation (5) of Yang-Mills fields. A consistent formulation

of the abelian gauge field  $A_{\mu\nu\rho}$  in the sum over histories approach requires that the field strength  $F$ , while non dynamical in the sense that it propagates no physical quanta, has support over a finite volume spacetime region, even in the absence of interactions, and gives rise to a Casimir vacuum pressure that is inversely proportional to the confinement volume. These results have been confirmed by an explicit computation of the vacuum expectation value of the energy-momentum tensor. With such results in hands, we have calculated the Wilson loop of the three-index potential coupled to a test spherical membrane. From the Wilson factor we have then extracted the static potential, Eq.(92), between pairs of opposite points on the membrane. The “volume law” encoded in Eq.(92) is a natural generalization of the well known “area law” for the static potential between two test charges (quarks) bound by a chromodynamic string. As a matter of fact, it may be useful to compare the result of equation (92) with the more familiar result for the Wilson loop of a quark-antiquark pair bounded by a string. In the latter case, the integration path is taken to be an elongated ( in the euclidean time direction ) rectangle of spatial side  $R$ . It is generally assumed that confinement is equivalent to

$$W \propto \exp(-\sigma A) \quad (93)$$

where  $A = TR$  is the area of the rectangle and  $\sigma$  is a constant with dimensions of (length squared)<sup>-2</sup>. From the definition (76) one extracts a linear potential between the two test quarks

$$V(R) = \sigma R. \quad (94)$$

The rising of the potential with the distance between charges corresponds to the fact that an increasing energy is necessary to separate them. In correspondence with Eq.(94) we found the expression (92) according to which the energy needed to separate antipodal points rises as  $R^3$ . This cubic law follows from the fact that the two charges under consideration are located on a spherical membrane rather than at the endpoints of an open string. Note that Eq.(92) and Eq.(94) describe the same kind of geometric behavior. In both cases the static potential is proportional to the “volume” of the manifold connecting the two test charges. In Eq.(94),  $R$  is essentially the “linear volume” of the string connecting the pair of test charges. In our case,  $R^3$  is proportional to the volume of the spherical membrane connecting the two antipodal points. Thus, we conclude that in the bag case, confinement is signaled by a “volume” law extending the string case area law. It has been noted elsewhere [11] that this is the exact counterpart, in four spacetime dimensions, of the situation encountered in the two dimensional Schwinger model that is widely believed to be the prototype model of quark confinement. The precise correspondence of the dynamics of the  $A_{\mu\nu\rho}$ -field coupled to quantum spinor fields and the dynamics of the Schwinger model will be the subject of a subsequent article in this series.

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